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Geometric flows and (some of) their physical applications*

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Abstract

The geometric evolution equations provide new ways to address a variety of non-linear problems in Riemannian geometry, and, at the same time, they enjoy numerous physical applications, most notably within the renormalization group analysis of non-linear sigma models and in general relativity. They are divided into classes of intrinsic and extrinsic curvature flows. Here, we review the main aspects of intrinsic geometric flows driven by the Ricci curvature, in various forms, and explain the intimate relation between Ricci and Calabi flows on Kähler manifolds using the notion of super-evolution. The integration of these flows on two-dimensional surfaces relies on the introduction of a novel class of infinite dimensional algebras with infinite growth. It is also explained in this context how Kac's K_2 simple Lie algebra can be used to construct metrics on S^2 with prescribed scalar curvature equal to the sum of any holomorphic function and its complex conjugate; applications of this special problem to general relativity and to a model of interfaces in statistical mechanics are also briefly discussed.

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The geometric evolution equations are parabolic systems that describe the deformation of metrics on Riemannian manifolds driven by their curvature in various forms. The continuous parameter t of the evolution is typically called time. These equations arise in a variety of non-linear problems in physics and mathematics and led to ground breaking results in recent years. They are naturally divided into classes of intrinsic and extrinsic curvature flows. The first class refers to deformations driven by the intrinsic Ricci curvature tensor on a manifold, whereas the second class refers to deformations of submanifolds embedded in higher dimensional spaces that evolve by their extrinsic curvature. Here, we will only be concerned with the mathematical structure and physical applications of intrinsic curvature flows, and review, apart from the basic facts, some recent results on their integrability in two dimensions, [1, 2, 3, 4], by focusing on the so called Ricci and Calabi flows. Selected applications to quantum field theory and general relativity will also be discussed among others. There is also a small number of new results and connections that are distributed in the text.

The subject of extrinsic curvature flows, which is equally interesting and much older, in certain respects, will not be included in this presentation. We only mention verbose, without formulae, the important role of the so called inverse mean curvature flow in general relativity. It was first introduced by Geroch, [5], in an attempt to prove positive energy theorems and examine the issue of naked singularities within Penrose's program, [6]. Profound results were obtained recently in this direction by proving the Riemannian Penrose inequality on general grounds, [7]; see also reference [8] for a more popular account. Another example of extrinsic flow is the so called mean curvature flow, which was first introduced as model for the motion of grain boundaries in an annealing piece of metal, [9], and then put on firm mathematical basis in a monograph by Brakke, [10], and the work of many others that followed. Such deformations are typically associated to surface tension forces of embedded submanifolds and the ruling equation is a gradient flow for the corresponding area functional. It turns out that the mean curvature flow can be identified with the renormalization group equation of Dirichlet sigma models away from conformality, [11], and then applied to various models of boundary quantum field theory of current interest, [12]. Further details about this interpretation, as well as generalizations with the Dirac-Born-Infeld action, will appear in a forthcoming publication, [13]. This ends the brief account of another interesting topic that will be excluded here.

The Ricci flow constitutes the prime example of intrinsic curvature flows. It is a system of second order non-linear parabolic equations for the components of the metric on any Riemannian manifold M, which undergo continuous deformations driven by the Ricci curvature tensor according to

$$\frac{\partial}{\partial t} g_{\mu\nu} = -R_{\mu\nu} \ . \tag{1}$$

It defines an infinite dimensional dynamical system in superspace, which consists of all possible metrics on a given Riemannian manifold, and as such it is very hard to analyze in all generality. The short time existence of its solutions is guaranteed by the parabolic form of the dynamics, but there might be singularities that arise at finite time, as in the

case of compact manifolds with strictly positive curvature metrics. In this case the space collapses and the flow becomes extinct. It is customary in the mathematics literature to work with the so called normalized Ricci flow using the mean scalar curvature $< R >$ on M,

$$\frac{\partial}{\partial t} g_{\mu\nu} = -R_{\mu\nu} + \frac{< R >}{\mathrm{dim} M} \; g_{\mu\nu} \; , \tag{2}$$

which, unlike the original form of the flow, preserves the total volume of space. Thus, it has a better chance to admit solutions that exist for sufficiently long time and converge asymptotically to constant curvature metrics, either positive or negative, depending on the topology of space. For this reason it has been employed as major new tool to address a variety of non-linear problems in Riemannian geometry, and, most notably, for the uniformization of manifolds in two and three dimensions. Of course, the normalized Ricci flow is no other than the unnormalized flow, since the two are related to each other by suitable time reparametrization and rescaling of the metric by a function of time.

The Ricci flow was first introduced in mathematics by Hamilton in the early eighties, [14], although the subject has much longer history in physics, where it appeared in the renormalization group studies of non-linear sigma models in two dimensions with fields taking values in a Riemannian manifold M. The original computation was performed by Polyakov who only considered the class of $O(n)$ sigma models with target space metric that of the round sphere on S^{n-1} for $n \geq 3$. The radius of the sphere is the only parameter of the theory, and, as such, it serves as the inverse of its coupling constant, i.e., $R \sim 1/g$. These two-dimensional models are not conformally invariant in the quantum regime, as it was found that the coupling constant runs, [15],

$$\frac{1}{\tilde{g}^2} = \frac{1}{g^2} + \frac{n-2}{4\pi} \mathrm{log} \frac{\tilde{\Lambda}}{\Lambda} \tag{3}$$

with respect to the world-sheet renormalization scale parameter ($\sim \Lambda^{-1}$), to lowest order in perturbation theory. As a result, the beta function is negative for all $n \geq 3$ and the theory becomes asymptotically free in the ultra-violet regime, thus providing a toy model for the asymptotic freedom that governs the ultra-violet behavior of the gauge theory of strong interactions in four space-time dimensions, [16, 17], and makes perturbation theory reliable for calculations. In more geometrical terms, the result corresponds to a particular solution of the (unnormalized) Ricci flow equation, which describes a uniformly contracting round sphere S^{n-1} with radius-square that diminishes linearly with $t = \mathrm{log} \Lambda^{-1}$; the flow starts from a very large (small curvature) sphere in the ultra-violet region $t \rightarrow -\infty$ and evolves continuously towards smaller sizes, until the sphere crashes to a point. Close to the singularity, however, the lowest order expression for the beta function is not reliable for higher order corrections and non-perturbative effects, in particular, may lead to a different physical interpretation, as they do.

The computation of the perturbative beta function was subsequently extended by Friedan to encompass two-dimensional non-linear sigma models with generalized coupling

given by the target space metric g of arbitrary Riemannian manifolds. It follows, [18],

$$\Lambda^{-1}\frac{\partial}{\partial\Lambda^{-1}}g_{\mu\nu} = -\beta(g_{\mu\nu}) = -R_{\mu\nu} + \cdots \ , \tag{4}$$

where the dots stand for higher order curvature corrections. The identification with the Ricci flow follows by setting t equal to the renormalization group time $\log\Lambda^{-1}$. Here, we will not be concerned with the form of higher order corrections and instanton effects to the beta function; the Ricci flow will be taken at face value, as in the mathematics literature. This calculation led to the development of the sigma model approach to string theory, since the requirement of conformal invariance for the world-sheet quantum field theory gives rise to vacuum Einstein equation in target space, $R_{\mu\nu} = 0$, plus higher order corrections if they are included. Further generalizations were also considered by appending other massless modes of the string, such as a dilaton and an anti-symmetric tensor field (torsion), to the action of the world-sheet sigma model. In this case one obtains a generalized system of beta function equations whose fixed points provide consistent backgrounds for string propagation. We only note here that the effect of the dilaton manifests as reparametrization along the Ricci flow, according to the more general equation

$$\frac{\partial}{\partial t}g_{\mu\nu} = -R_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \ , \tag{5}$$

when ξ is a gradient vector field. Finally, the renormalization group trajectories, away from fixed points, have been used to provide an off-shell formulation of the problem of tachyon condensation in closed string theory that accounts for the decay of an unstable vacuum to another more stable vacuum.

The presence of torsion affects the formation of singularities along the renormalization group flow and deserves systematic study in the future. Here, we will not be concerned with it because most of our results will be confined to models with two-dimensional target spaces, where there can be no perturbative contributions to the beta functions due to torsion; any contributions of this kind appear through the corresponding field strength, which is a 3-form, that vanishes identically in two dimensions. However, one may also add instanton contributions to the beta function that capture the effect of topological torsion in two-dimensional target space, as in the case of the $O(3)$ sigma model with a θ-term. It is well known by now that at $\theta = \pi$ the model becomes massless and its infrared limit is governed by the $SU(2)_1$ Wess-Zumino-Witten model, i.e., a free boson on a circle with its radius held fixed at the enhanced $SU(2)$ symmetry point, [19]. This result also finds important applications to the explanation of certain aspects of the quantum Hall effect (see, for instance, [20], and references therein). It may serve as starting point for working mathematicians to explore the resolution of singularities of the Ricci flow by instantons and study their implications to geometry. We intend to return to these issues elsewhere, since they have been omitted from our work so far.

There is already vast literature in mathematics concerning the qualitative behavior of solutions and the formation of singularities along the (normalized) Ricci flow in low dimensions (see, for instance, [21, 22, 23], and references therein). These studies play

important role in three dimensions, in particular, and have been advanced towards a proof of Thurston's geometrization conjecture by the recent work of Perelman, [24]. It is a subject of great activity in mathematics, since many steps of the proposed proof are being analyzed and cross checked by the experts; see, for instance, [25], for a more popular account. Apart from these general considerations there are also some simple solutions of the Ricci flow in low dimensions that were obtained by consistent truncation to mini-superspace sectors and depend on a small number of moduli. In two dimensions, they include the sausage model as an axially symmetric deformation of the $O(3)$ sigma model, [26], and the decay of a cone C/Z_n to the plane, [27]. The latter solution, which generalizes the fundamental (Gaussian) solution of the heat flow equation, exhibits a delta function singularity in the curvature at some initial time, which dissipates completely in the infra-red region, as $t \rightarrow +\infty$. It also serves as a model to study localized tachyon condensation in closed sting theory. In three dimensions, there is the $O(4)$ sigma model generalization of the sausage deformation, [28], as well as various examples of mini-superspace deformations within the Bianchi classification of allowed isometries, [29]. There are also numerical studies of different trajectories that depend on specific initial data, which provide preliminary evidence for critical behavior along the Ricci flow, [30], as in the problem of gravitational collapse. In fact, this analogy will become stronger if one finds an off-shell description of the gravitational collapse in four space-time dimensions in terms of the Ricci flow on three-dimensional Riemannian manifolds.

Next, we turn the discussion to the Calabi flow, which, unlike the Ricci flow, is only defined on Kähler manifolds with complex coordinates $(z^a, \bar{z}^a)$. It has the following form, [31],

$$\frac{\partial}{\partial t} g_{a\bar{b}} = \frac{\partial^2 R}{\partial z^a \partial \bar{z}^b} \ , \tag{6}$$

using the scalar Ricci curvature, which in complex notation is

$$R = g^{a\bar{b}} R_{a\bar{b}} = -g^{a\bar{b}} \partial_a \bar{\partial}_b \mathrm{log}(\mathrm{det} g) \ . \tag{7}$$

The Calabi flow is a fourth order equation for the components of the metric, and, thus, more complicated to study in all generality as compared to the second order Ricci flow. In fact, although several simple solutions are explicitly known for the Ricci flow, there are hardly any examples available in the literature for trajectories of the Calabi flow. In other words, it appears to be difficult to device mini-superspace models of the Calabi flow for which all relevant deformations can be consistently truncated to a finite number of moduli. Both Kähler-Ricci and Calabi flows induce deformations of the metric in a fixed cohomology class, but the total volume of space is preserved in the latter case, unlike the unnormalized Ricci flow. Critical points of the Calabi flow are called extremal metrics. They only exit under certain technical conditions and often describe constant curvature metrics on a given Kähler manifold; see, for instance, [32], for reviews of the subject. The restriction of geometric evolution equations to Kählerian spaces is a great relief for the system of induced deformations, since there is only one function, namely the Kähler potential K, that determines the form of the metric as $g_{a\bar{b}}(z, \bar{z}) = \partial_a \bar{\partial}_b K(z, \bar{z})$. Yet, there

are several applications to geometry that include new proofs of uniformization theorems and the investigation of some open conjectures for the curvature of Kähler manifolds.

In the following, we restrict attention to geometric evolutions on two-dimensional surfaces and write the metric in a system of conformally flat (Kähler) coordinates $(z, \bar{z})$,

$$ds_{\rm t}^2 = 2e^{\Phi(z,\bar{z};t)}dzd\bar{z} \ . \tag{8}$$

Then, the two different flows assume the simple form

$$\text{Ricci flow} \quad : \qquad \frac{\partial \Phi}{\partial t} = \Delta \Phi \ , \tag{9}$$

$$\text{Calabi flow} \quad : \qquad \frac{\partial \Phi}{\partial t} = -\Delta \Delta \Phi \ , \tag{10}$$

where Δ is the Laplace-Beltrami operator on the surface,

$$\Delta = e^{-\Phi}\partial\bar{\partial} \ . \tag{11}$$

These are non-linear differential equations of second and fourth order, respectively, that resemble the heat flow equation on the surface due to the parabolic character of their dependence on the time variable t. As such, they exhibit dissipative behavior in time, but, as will be seen later, they turn out to be integrable in space with the aid of appropriately chosen infinite dimensional Lie algebras that incorporate the deformation variable t into their defining relations.

It should be noted at this point, as side remark, that the two-dimensional geometric flows arise as limiting cases of more general diffusion equations on surfaces. First, let us consider the porous medium equation

$$\frac{\partial u}{\partial t} = \partial\bar{\partial}u^m \ , \tag{12}$$

where m is an arbitrary (possibly fractional) exponent. The limit $m \to 0$ is well defined provided that t is also rescaled by m so that mt remains finite. Then, in the new time variable, the evolution becomes (see, for instance, [33])

$$\frac{\partial u}{\partial t} = \lim_{m \to 0} \frac{1}{m}\partial\bar{\partial}u^m = \partial\bar{\partial}{\rm log}u \ , \tag{13}$$

which is identical to the Ricci flow provided that $u = e^{\Phi}$. Likewise, there is a fourth order diffusion equation (with no name, to the best of our knowledge) of the form

$$\frac{\partial u}{\partial t} = -\partial\bar{\partial}(u^{-1}\partial\bar{\partial}u^m) \ , \tag{14}$$

whose $m \to 0$ limit yields in a similar fashion the equation

$$\frac{\partial u}{\partial t} = -\partial\bar{\partial}(u^{-1}\partial\bar{\partial}{\rm log}u) \ , \tag{15}$$

i.e., the two-dimensional Calabi flow for the variable $u = e^{\Phi}$. Some properties of the geometric evolution equations are inherited from these more general diffusion processes, although it is not clear to us whether the group theoretical methods used later for their integration also extend to the equations with arbitrary values of the exponent m.

A physical context for the Calabi flow on two-dimensional surfaces with spherical topology is provided by the class of Robinson-Trautman metrics in general relativity. These are radiative metrics in four space-time dimensions of the form, [34, 35]

$$ds^2 = 2r^2 e^{\Phi(z,\bar{z},t)} dz d\bar{z} - 2dtdr - H(z, \bar{z}, r, t)dt^2 \tag{16}$$

and they represent spherical gravitational waves in vacuum. The time component of the metric takes the form

$$H = r\frac{\partial \Phi}{\partial t} - \Delta\Phi - \frac{2m}{r} , \tag{17}$$

where m is a mass parameter that can be taken to be constant. In this context, Einstein's equations amount to a single differential equation for the conformal factor of closed surfaces with constant r and t, which reads

$$\Delta\Delta\Phi + 3m\frac{\partial \Phi}{\partial t} = 0 . \tag{18}$$

When $m > 0$, it is identical to the Calabi flow with deformation parameter t given by the retarded time, [36]. Without loss of generality, one may set $3m = 1$ in appropriate units. In this case, the most general solution describes type II space-times that evolve towards the Schwartzschild metric (see [37] and references therein for earlier work on the subject). Thus, it becomes very important to explore the exact solvability of this particular dynamical sector of Einstein's equation in terms of the two-dimensional Calabi flow. Higher dimensional generalizations of this flow have not yet appeared in physical problems to the best of our knowledge.

The time evolution has the tendency to dissipate away any curvature perturbations of the canonical metric on the two-dimensional surface. In fact, after sufficiently long time, the metrics deform towards the constant curvature metric on the surface, which for the case of S^2 is $ds^2 = R_0^2(d\theta^2 + \sin^2\theta d\phi^2)$ in a system of spherical coordinates (θ, ϕ). Axially symmetric deformations of a round sphere with radius R_0 are conveniently described in the form

$$ds_{\mathrm{t}}^2 = R_0^2[1 + \epsilon_l(t)P_l(\cos\theta)] \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{19}$$

using Legendre polynomials $P_l(\cos\theta)$ with $l \geq 2$. These deformations preserve the volume of the space. Then, to linear order in the perturbation parameters $\epsilon_l(t)$, the Calabi flow yields the following evolution

$$\epsilon_l(t) = \epsilon_l(0)\exp\left(-\frac{t}{4R_0^4}l(l^2 - 1)(l + 2)\right) \tag{20}$$

that approximates well the asymptotic behavior of the full non-linear evolution as $t \to +\infty$. The result implies that the perturbations are damped exponentially fast for all $l \geq 2$

and the configuration settles down to that of a round metric on S^2. The damping is faster for higher values of l. In the context of Robinson-Trautman metrics, these perturbations correspond to linearized multi-pole gravitational radiation, [38], as the geometry tends asymptotically to the Schwartzschild metric in Eddington-Finkelstein coordinates.

Direct comparison with the dissipative properties of the Ricci flow on S^2 requires making use of the normalized (rather than the unnormalized) form of the evolution

$$\frac{\partial \Phi}{\partial t} = \Delta \Phi + \frac{1}{R_0^2} \tag{21}$$

so that the volume of the space is preserved. Then, the ansatz of infinitesimal axially symmetric deformations can be consistently implemented, and, to linear order in $\epsilon_l(t)$, it follows that

$$\epsilon_l(t) = \epsilon_l(0) \mathrm{exp} \left(-\frac{t}{2R_0^2}(l-1)(l+2) \right) . \tag{22}$$

The perturbations are also damped exponentially fast, but at slower pace as compared to the fall off rates of the Calabi flow. Their specific dependence on l reflects the difference in the order of the corresponding differential equations. Also, in either case, it is observed that the decay of perturbations depends on the radius R_0 and it is bigger for smaller spheres.

There is a formal relation between Ricci and Calabi flows on Kähler manifolds, which is seen by taking the time derivative of the Ricci flow in complex coordinates, [4],

$$\frac{\partial^2}{\partial t^2} g_{a\bar{b}} = -\frac{\partial}{\partial t} R_{a\bar{b}} = \partial_a \bar{\partial}_b \frac{\partial}{\partial t} (\log \, \mathrm{det} g) = -\partial_a \bar{\partial}_b R \ . \tag{23}$$

Thus, if the second derivative of the Kähler metric with respect to the Ricci time t_{R} is identified with minus its first derivative with respect to the Calabi time t_{C}, i.e.,

$$\frac{\partial^2}{\partial t_{\mathrm{R}}^2} = -\frac{\partial}{\partial t_{\mathrm{C}}} \ , \tag{24}$$

the two flows will be formally the same. This observation relates two parabolic equations of second and fourth order and it is reminiscent of the way to extract the square root of the Schrödinger equation in supersymmetric quantum mechanics. It will be particularly useful for the parallel treatment of the two flows on two dimensional surfaces.

The integration of Ricci and Calabi flows beyond the linearized approximation relies on the use of appropriately chosen infinite dimensional algebras that enable to cast the equations into zero curvature form in two dimensions. The treatment of the two flows will be similar, as they both admit Toda field theoretic interpretation with gauge connections taking values in the class of (super)-continual Lie algebras for specific choices of the Cartan operator. In preparation of this, it is useful to implement the formal relation between the Ricci and Calabi flows on Kähler manifolds using the notion of super-evolution. In particular, let us introduce an anti-commuting variable θ, with $\theta^2 = 0$, as the supersymmetric partner of the deformation variable t, and write down the super-evolution Ricci type equation, [4],

$$\mathcal{D}_T \mathrm{exp} \mathcal{F} = \partial \bar{\partial} \mathcal{F} \ , \tag{25}$$

where

$$\mathcal{F}(z, \bar{z}; T) = \Phi(z, \bar{z}; t) + \theta\Psi(z, \bar{z}; t) \tag{26}$$

is a superfunction in $R^{1|1}$ superspace with coordinates $T = (t, \theta)$ and $\mathcal{D}_T$ is the associated super-derivative operator

$$\mathcal{D}_T = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t} \tag{27}$$

that satisfies the relation $\mathcal{D}_T^2 = -\partial/\partial t$. The components Φ and Ψ are ordinary (bosonic) functions of t that are also taken to depend on the complex coordinates $(z, \bar{z})$ on the surface. Then, in terms of components, the super-evolution equation for $\mathcal{F}$ reads

$$e^{\Phi}\Psi = \partial\bar{\partial}\Phi \ , \qquad \frac{\partial e^{\Phi}}{\partial t} = -\partial\bar{\partial}\Psi \ , \tag{28}$$

which leads to the two-dimensional Calabi flow for Φ by eliminating the field Ψ. Thus, taking the square root of the time derivative operator, allows to connect the two distinct classes of geometric deformations of second and fourth order, respectively, via super-evolution, as was anticipated before on general grounds.

The algebraic framework that is appropriate to use for integrating the Ricci and Calabi flows in two dimensions is provided by the class of (super)-continual Lie algebras with basic system of commutation relations, [39],

$$[H(\varphi), \ X^{\pm}(\psi)] = \pm X^{\pm}((K\varphi)\psi) \ , \tag{29}$$

$$[X^{+}(\varphi), \ X^{-}(\psi)] = H(S(\varphi\psi)) \ , \tag{30}$$

$$[H(\varphi), \ H(\psi)] = 0 \ . \tag{31}$$

The elements H and $X^{\pm}$ generate the local part of the algebra, whereas all other generators can be obtained, in principle, by taking successive commutators of these basic elements. The smearing functions can be either ordinary functions of the continuous variable t, in which case the corresponding Lie algebra is called continual, or superfunctions of the super-variable T, in which case we will refer to it as super-continual Lie algebra, [4]. In either case, the algebras are bosonic, infinite dimensional, and they are solely characterized by the choice of operators K and S that act linearly on the space of smearing (super)functions. For the case of invertible operators, it is possible to make the canonical choice $S = 1$ by redefining K as $\tilde{K} = KS$. This choice will be adopted in the following, unless it is explicitly stated otherwise, and $\tilde{K}$ will be called Cartan operator; the tilde will also be dropped for simplicity.

Alternatively, one may consider a system of basic generators $H(t)$ and $X^{\pm}(t)$ which are related to the smeared ones as

$$A(\varphi) = \int A(t)\varphi(t)dt \tag{32}$$

when $A(t)$ is smeared with any smooth function $\varphi(t)$ with compact support. In this formulation, the continual Lie algebra is characterized by the choice of Cartan kernel

$K(t,t')$, which is in general a distribution. For super-continual Lie algebras, the generators can be alternatively thought to depend on T, so that $A(T) = A_0(t) + \theta A_1(t)$, and their smearing is simply defined by integration in $R^{1|1}$ superspace against any suitably chosen superfunction $\mathcal{F}(T) = \varphi_0(t) + \theta\varphi_1(t)$, as

$$A(\mathcal{F}) = \int A(T)\mathcal{F}(T)dT = A_0(\varphi_1) + A_1(\varphi_0) \ , \tag{33}$$

using the relations $\int d\theta = 0$ and $\int \theta d\theta = 1$. Thus, it is possible to work out the form of the basic commutation relations of the Lie algebra in terms of bosonic components H_i and $X_i^{\pm}$ with $i = 0, 1$ in smeared or unsmeared form, if desired.

For any given choice of Cartan operator K there is an associated Toda equation for the field $\Phi(z, \bar{z}; t)$

$$\partial\bar{\partial}\Phi = K(e^{\Phi}) \tag{34}$$

that admits zero curvature representation in two dimensions,

$$[\partial + A_+(z, \bar{z}), \ \bar{\partial} + A_-(z, \bar{z})] = 0 \ , \tag{35}$$

using gauge connections $A_{\pm}$ with values in the corresponding continual Lie algebra. Indeed, the particular choice

$$A_+ = H(g) + X^+(1) \ , \qquad A_- = X^-(e^{\Phi}) \tag{36}$$

amounts to the following system of equations

$$\bar{\partial}g = e^{\Phi} \ , \qquad \partial\Phi = K(g) \ , \tag{37}$$

which yield the Toda field equation above after eliminating g. In the case of super-continual Lie algebras the framework is the same provided that both Φ and g are superfunctions of T rather than ordinary functions of t.

It is becoming clear now that the two-dimensional Ricci flow admits the following Toda field theory interpretation

$$\frac{\partial e^{\Phi}}{\partial t} = \partial\bar{\partial}\Phi \ : \qquad K = \frac{\partial}{\partial t} \ , \tag{38}$$

using a continual Lie algebra with Cartan operator $K = \partial/\partial t$, [1, 2], whereas for the Calabi flow a similar interpretation follows easily from its equivalent description as super-evolution equation

$$\mathcal{D}_T \mathrm{exp}\mathcal{F} = \partial\bar{\partial}\mathcal{F} \ : \qquad K = \mathcal{D}_T = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t} \ , \tag{39}$$

using a super-continual Lie algebra with odd Cartan operator $K = \mathcal{D}_T$ equal to the square root of $-\partial/\partial t$, [4]. In either case, the main idea of this formulation is to treat the coordinates $(z, \bar{z})$ and t in uneven way and incorporate the deformation variable into the defining relations of an infinite dimensional (super)-continual Lie algebra. Then, the resulting Toda field theory interpretation of the flows allows for their zero curvature

formulation in two dimensions, and, eventually, for their (formal) integration by group theoretical methods, as outlined below.

The general solution of Toda field equations with Cartan operator K is obtained by straightforward generalization of the group theoretical construction that is available for ordinary Toda systems, [39]. Let us introduce a highest weight state $|t>$ that depends on the continuous variable t and satisfies the conditions

$$X^+(t')|t>=0 \ , \qquad <t|X^-(t')=0 \ , \qquad H(t')|t>=\delta(t-t')|t> \tag{40}$$

and $<t|t>=1$. The existence of such state is not guaranteed for continual Lie algebras, but, nevertheless, it will be formally introduced for all K. We also consider a one-parameter family (parametrized by t) of two-dimensional free fields given by the sum of an arbitrary holomorphic function and its complex conjugate,

$$\Phi_0(z,\bar{z};t)=f(z;t)+\bar{f}(\bar{z};t) \ , \tag{41}$$

and define the path-ordered exponentials

$$\begin{aligned} M_+(z) &= \mathcal{P}\mathrm{exp}\left(\int^z dz'\int dt' e^{f(z';t')}X^+(t')\right) , \\ M_-(\bar{z}) &= \mathcal{P}\mathrm{exp}\left(\int^{\bar{z}} d\bar{z}'\int dt' e^{\bar{f}(\bar{z}';t')}X^-(t')\right) \end{aligned} \tag{42}$$

obtained by suitable smearing of the generators $X^\pm$ with the holomorphic and anti-holomorphic components of Φ_0, respectively. Then, the Toda field configurations admit, in general, the following free field representation

$$\Phi(z,\bar{z};t)=\Phi_0(z,\bar{z};t)-K\left(\log <t|M_+^{-1}(z)M_-(\bar{z})|t>\right) \ . \tag{43}$$

The expression above gives rise to an infinite power series by expanding the path-ordered exponentials $M_\pm$ and computing the expectation value

$$\begin{aligned} <t|M_+^{-1}M_-|t>= 1+\sum_{n=1}^{\infty}(-1)^n\int^z dz_1\cdots\int^{z_{n-1}}dz_n\int^{\bar{z}}d\bar{z}_1\cdots\int^{\bar{z}_{n-1}}d\bar{z}_n \times \\ \times\int\prod_{i=1}^n dt_i\int\prod_{i=1}^n dt_i'\mathrm{exp}f(z_i;t_i)\mathrm{exp}\bar{f}(\bar{z}_i;t_i')D_t^{\{t_1,t_2,\cdots,t_n;t_n',\cdots,t_2',t_1'\}} \end{aligned} \tag{44}$$

with $z\geq z_1\geq\cdots\geq z_n$ and likewise for the $\bar{z}$'s. All information about the algebra is fully encoded into the elements

$$D_t^{\{t_1,t_2,\cdots,t_n;t_1',t_2',\cdots,t_n'\}}=<t|X^+(t_1)X^+(t_2)\cdots X^+(t_n)X^-(t_n')\cdots X^-(t_2')X^-(t_1')|t> \tag{45}$$

that can be computed recursively by shifting X^+'s to the right and X^-'s to the left. Provided that the infinite series converges, the result is interpreted as a formal expansion of the non-linear configuration Φ around the free field Φ_0 in powers of $\mathrm{exp}\Phi_0$.

According to this framework, the general solution of the two-dimensional Ricci flow follows by specialization to the continual Lie algebra with Cartan operator $K=\partial/\partial t$. It is

an example of a novel infinite dimensional algebra with anti-symmetric Cartan kernel that exhibits infinite growth beyond its local part, [39]. Particular details of the calculation and some simple examples can be found in the literature, [1], where the validity of the group theoretical integration has also been tested using exact mini-superspace models of axially symmetric deformations of constant curvature metrics. For example, axially symmetric configurations with free fields of the form $\Phi_0 = c \cdot (z + \bar{z}) + d(t)$ admit the expansion in powers of $\exp\Phi_0$,

$$\Phi(z, \bar{z}; t) = \Phi_0 + \frac{1}{(1!\ c)^2}\partial_t e^{\Phi_0} + \frac{1}{(2!\ c^2)^2}\partial_t \left(e^{\Phi_0}\partial_t e^{\Phi_0}\right) + \mathcal{O}(e^{3\Phi_0}) \ . \tag{46}$$

Likewise, the general solution of the two-dimensional Calabi flow (in superfield form) follows by specialization to the super-continual Lie algebra with odd Cartan operator $K = \mathcal{D}_T$, as

$$\mathcal{F}(z, \bar{z}; T) = \mathcal{F}_0(z, \bar{z}; T) - K\left(\log < T|M_+^{-1}(z)M_-(\bar{z})|T >\right) . \tag{47}$$

The corresponding normalized highest weight state $|T >= |t >_0 + \theta|t >_1$ is introduced here using the smeared form of the defining relations,

$$X^+(\varphi)|\psi >= 0 \ , \qquad < \psi|X^-(\varphi) = 0 \ , \qquad H(\varphi)|\psi >= |\varphi\psi >, \tag{48}$$

for all superfunctions φ and ψ, and $|\psi >= |\psi_1 >_0 + |\psi_0 >_1$. Also, the free superfield is decomposed into sum of an arbitrary holomorphic superfunction and its complex conjugate, which are subsequently used to smear $X^\pm(T)$ in the corresponding expressions of the path-ordered exponential $M_\pm$. It turns out that axially symmetric deformations which correspond to free field configurations $\mathcal{F}_0 = \Phi_0 + \theta\Psi_0$ with $\Phi_0 = c \cdot (z + \bar{z}) + d(t)$ and $\Psi_0 = a \cdot (z + \bar{z}) + b(t)$, admit the following power series expansion

$$\Phi = \Phi_0 + \frac{1}{(1!\ c)^2}\left(\Psi_0 - \frac{2a}{c}\right)e^{\Phi_0} - \frac{1}{(2!\ c^2)^2}\left(e^{\Phi_0}\partial_t e^{\Phi_0} - \left(\Psi_0^2 - \frac{4a}{c}\Psi_0 + \frac{7a^2}{2c^2}\right)e^{2\Phi_0}\right) \tag{49}$$

plus higher terms of order $\mathcal{O}(e^{3\Phi_0})$. We refer the reader to the literature for further technical details, [4].

We are turning now the discussion to a special topic related to static configurations of the two-dimensional Calabi flow, namely $\partial\bar{\partial}R = 0$, which amount to the following fourth order equation for the conformal factor,

$$\Delta\Delta\Phi = 0 \ . \tag{50}$$

Constant curvature metrics arise as special solutions and they correspond to the Liouville equation, $\Delta\Phi = const.$, which admits zero curvature formulation in terms of $SL(2)$-valued gauge connections; its general solution is described using an arbitrary holomorphic function (and its complex conjugate) by specializing the integration scheme for Toda systems to the algebra with Cartan operator $K = 1$. Constant curvature metrics also describe the static configurations of the normalized Ricci flow in two dimensions. More

generally, however, solutions of the fourth order equation correspond to conformally flat metrics with prescribed scalar curvature equal to the sum of an arbitrary holomorphic function and its complex conjugate, i.e.,

$$\Delta\Phi = \psi(z) + \bar{\psi}(\bar{z}) \ . \tag{51}$$

A particularly simple solution, in fact, the only known example of this kind with $\Delta\Phi = z + \bar{z}$, is

$$e^{\Phi} = \frac{3}{(z+\bar{z})^3} \ . \tag{52}$$

A well known physical framework for this equation is provided by the class of type III Robinson-Trautman metrics in general relativity, i.e., metrics with mass parameter $m = 0$, [34, 35].

An alternative description of the equation $\Delta\Delta\Phi = 0$ is obtained in the form, [4],

$$\partial\bar{\partial}\mathcal{F} = \frac{\partial}{\partial\theta}e^{\mathcal{F}} \tag{53}$$

using a superfunction $\mathcal{F}(z, \bar{z}; \theta) = \Phi(z, \bar{z}) + \theta\Psi(z, \bar{z})$ with components related to each other as $\Psi = \Delta\Phi$. This formulation is advantageous for the formal integration of the equation by group theoretical methods, as in Toda systems. It turns out that the appropriate choice is the super-continual Lie algebra with $K = 1$ and $S = \partial/\partial\theta$ in the nomenclature introduced earlier. There is no canonical choice of Cartan operator in this special case, since $S^2 = 0$; also the dependence of the algebra generators upon t is superfluous and will be dropped out completely, thus leaving only their dependence on the Grassmann variable θ. For later reference, it is useful to present the form of the commutation relations among the basic generators in terms of components,

$$\begin{aligned} &[H_1, \ X_0^{\pm}] = \pm X_0^{\pm} \ , \qquad [X_1^{\pm}, \ X_0^{\mp}] = \pm H_1 \ , \\ &[H_0, \ X_1^{\pm}] = \pm X_0^{\pm} \ , \qquad [H_1, \ X_1^{\pm}] = \pm X_1^{\pm} \ , \end{aligned} \tag{54}$$

whereas the rest are trivial,

$$\begin{aligned} &[X_0^{+}, \ X_0^{-}] = 0 \ , \qquad [X_1^{+}, \ X_1^{-}] = 0 \ , \\ &[H_0, \ X_0^{\pm}] = 0 \ , \qquad [H_i, \ H_j] = 0 \ . \end{aligned} \tag{55}$$

It can be verified that the zero curvature condition for the gauge connections

$$A_+ = fH_0 + gH_1 + X_1^+ \ , \qquad A_- = \Psi e^{\Phi} X_0^- + e^{\Phi} X_1^- \tag{56}$$

amounts to $\Delta\Delta\Phi = 0$ by eliminating all other functions but Φ from the resulting system of equations.

The generators H_1 and $X_i^{\pm}$ form the local part of Kac's K_2 simple Lie algebra, as can be seen by setting

$$\begin{aligned} &e_0 = \frac{1}{\sqrt{2}}(X_0^+ + X_1^+) \ , \qquad e_1 = \frac{1}{\sqrt{2}}(X_0^+ - X_1^+) \ , \\ &f_0 = \frac{1}{\sqrt{2}}(X_0^- + X_1^-) \ , \qquad f_1 = \frac{1}{\sqrt{2}}(-X_0^- + X_1^-) \ . \end{aligned} \tag{57}$$

Indeed, by also denoting $H_1 = h$, it follows that

$$[e_i,\ f_j] = \delta_{ij} h \ , \qquad [h,\ e_i] = e_i \ , \qquad [h,\ f_i] = -f_i \ , \tag{58}$$

which are the defining commutation relations of K_2 algebra with infinite growth, [40]; see also reference [3] for an alternative algebraic formulation, as well as reference [41] for earlier work on the subject using prolongation methods.

Finally, it should be noted that although the zero curvature formulation of the problem requires the use of the additional element H_0, its integration by group theoretical methods employs only the K_2 subalgebra. Detailed analysis of the Toda system with $K = 1$ and $S = \partial/\partial\theta$ shows that the general solutions assumes the final form

$$\Phi(z, \bar{z}) = \Phi_0(z, \bar{z}) - \log\left(< 0|M_+^{-1}(z)M_-(\bar{z})|0 >\right) \tag{59}$$

where $|0>$ is a (formal) vacuum state of the K_2 algebra,

$$X_i^+|0>= 0 \ , \qquad < 0|X_i^- = 0 \ , \qquad H_1|0>= |0> \ , \tag{60}$$

and $M_\pm$ are the path-ordered exponentials

$$\begin{aligned} M_+(z) &= \mathcal{P}\exp\left(\int^z dz' e^{f(z')}[X_1^+ + \psi(z')X_0^+]\right) , \\ M_-(\bar{z}) &= \mathcal{P}\exp\left(\int^{\bar{z}} d\bar{z}' e^{\bar{f}(\bar{z}')}[X_1^- + \bar{\psi}(\bar{z}')X_0^-]\right) \end{aligned} \tag{61}$$

expressed in terms of the free fields $\Phi_0(z, \bar{z}) = f(z) + \bar{f}(z, \bar{z})$ and $\Psi_0(z, \bar{z}) = \psi(z) + \bar{\psi}(\bar{z})$. In geometrical terms, $\Psi_0(z, \bar{z})$ prescribes the curvature of the two-dimensional metric in question, as $\Delta\Phi = \Psi_0 = -R$, whereas Φ_0 will parametrize the general solution. In the simple case of axially symmetric metrics with $\Phi_0 = c \cdot (z + \bar{z}) + d$ and $\Psi_0 = a \cdot (z + \bar{z}) + b$, explicit calculation yields the power series expansion of the solution

$$\Phi = \Phi_0 + \frac{e^{\Phi_0}}{(1!\ c)^2}\left(\Psi_0 - \frac{2a}{c}\right) + \frac{e^{2\Phi_0}}{(2!\ c^2)^2}\left(\Psi_0^2 - \frac{4a}{c}\Psi_0 + \frac{7a^2}{2c^2}\right) + \mathcal{O}(e^{3\Phi_0}) \ . \tag{62}$$

Further details can also be found in the published work [4].

At this point we mention one more occurrence of the equation $\Delta\Delta\Phi = 0$ in the context of statistical mechanics. Recall first another fourth order parabolic equation of the form

$$\frac{\partial}{\partial t} e^\Phi = -\partial\bar{\partial}\left(e^\Phi \partial\bar{\partial}\Phi\right) , \tag{63}$$

which is a variant of the Calabi flow in that it involves the factor e^Φ rather than $e^{-\Phi}$ on its right-hand side; as such, it does not have a natural geometric interpretation. This equation, or better to say a one-dimensional reduction of it, arose while studying properties of interfaces between stationary phases of the two-dimensional (unbiased) Toom model, which are not described by equilibrium Gibbs ensembles, [42]. It has the remarkable property to admit a fundamental solution of Gaussian type,

$$e^{\Phi(z,\bar{z};t)} = \frac{1}{2\pi\sigma(t)}\exp\left(-\frac{z\bar{z}}{2\sigma(t)}\right) \ , \tag{64}$$

with $\sigma(t) = \sqrt{t/2}$. It is similar to the fundamental solution of the second order heat equation, but differs from it in the t-dependence of $\sigma(t)$ that spreads as $\sqrt{t}$ and not t. The Calabi flow does not admit such solutions. However, static solutions of this equation are identical to static solutions of the Calabi flow, as the two are related by flipping the sign of $\Phi(z, \bar{z})$, and they are also characterized by the fourth order equation $\Delta\Delta\Phi = 0$. It will be interesting to examine further the physical manifestation of K_2 algebra in such models of statistical mechanics.

In conclusion, we have reviewed the main aspects of geometric evolution equations as they arise in physics and mathematics with special emphasis on Ricci and Calabi flows in two dimensions. These are non-linear equations that are interrelated by methods of supersymmetric quantum mechanics and exhibit rich algebraic structure that enables us to cast them into zero curvature form. The infinite dimensional algebras that serve this purpose incorporate the deformation variable t into their defining system of commutation relations and have infinite growth. It will be interesting in this context to consider further generalizations in low dimensions by including the effect of torsion in the corresponding geometric flows as well as non-perturbative effects generated by instantons, as in the quantum field theory of non-linear sigma models with Wess-Zumino-Witten terms and topological θ-terms, respectively. Also, it will be interesting to examine the algebraic manifestation of entropy functionals for these flows in an attempt to study more systematically, and from a physics perspective, the new mathematical structures that were encountered above.

Finally, other classes of geometric evolution equations, in particular those related to extrinsic curvature flows, deserve more attention in the future in view of their applications to quantum field theory and general relativity. Several applications to other branches of physics have been excluded from this presentation due to space limitations. There is also the hope that many more parabolic equations can be treated by methods similar to those developed here. Last, but not least, we mention the possible use of geometric evolution equations, such as the Ricci flow in d spatial dimensions, for developing an off-shell formulation of the problem of gravitational collapse in $d + 1$ space-time dimensions.

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